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NOTE

Stability Analysis of the Euler–Poisson Equations

INTRODUCTION

This paper is devoted to the numerical stability analysis of a one-dimensional plasma model, in the fluid mechanics theory approach. We shall thus consider that all individual particle effects are neglected and that only the fluid motion is taken into account. The plasma evolution is described by the Euler equations, density, and momentum conservation laws where the interactions between charged particles appear as a force in the right-hand side of the equations. For more details on this approach, see [1]. The description of the electromagnetic field evolution is given by the Maxwell equations. We shall make several simplifying hypotheses to study an electrostatic model.

After setting our framework, we study the stability of the numerical methods used and obtain a relatively strong necessary condition to ensure the convergence of the scheme. Namely the full implicitness of the electric field is necessary for the sake of stability. It is surprising in a sense, because a particles method, used to solve the collisionless Boltzmann equation, coupled with a Poisson equation in a similar physical parameters range does not impose such a strong stability condition. In the last section, we give a numerical comparison between several ways to introduce the contribution of the Poisson equation in our numerical scheme.

I. DESCRIPTION OF THE MODEL AND LINEARIZATION

We assume the following hypothesis: the plasma is composed of two interpenetrating fluids, the ions and the electrons (this is for simplicity; an extension to more species is straightforward). We neglect collision effects and viscosity. We suppose that the ions are fixed in space in a uniform distribution and that the electrons are in an isothermal evolution so that their evolution is described by a unique thermodynamical variable, the density. We suppose next that electrons have a thermal energy, so that a pressure term will appear in the Euler equations.

In order to derive a one-dimensional electrostatic model, we assume that the plasma is infinite in extension and that the electron motions occur only in the x direction (see Fig. 1). Next, we suppose that there is no magnetic field; thus the electric field derives from a potential which satisfies the only Maxwell equation that does not involve the magnetic field: the Poisson equation. Finally, we obtain the following nonlinear system of partial differential equations, for x belonging to $[0, L]$,

$$\partial_t n + \partial_x nu = 0 \tag{1}$$

$$\partial_t nu + \partial_x(nu^2 + p) = -(e/m)\, nE \tag{2}$$

$$E = -\nabla\varphi \tag{3}$$

$$-\Delta\varphi = e/\varepsilon_0(n_i - n), \tag{4}$$

where n, u, p denote, respectively, the density, the velocity, and the pressure of the electrons fluid, e and m are the charge and mass of the electrons, E is the electric field, φ the potential associated to E and n_i is the ion density. The pressure term is given by

$$p = nV_T^2, \tag{5}$$

where V_T, the thermal velocity is defined by $V_T^2 = KT/m$; KT is the thermal energy of the electrons and is positive. The boundary conditions on Poisson equation (4), are $\varphi(0) = 0$ and $\partial_x\varphi(L) = 0$.

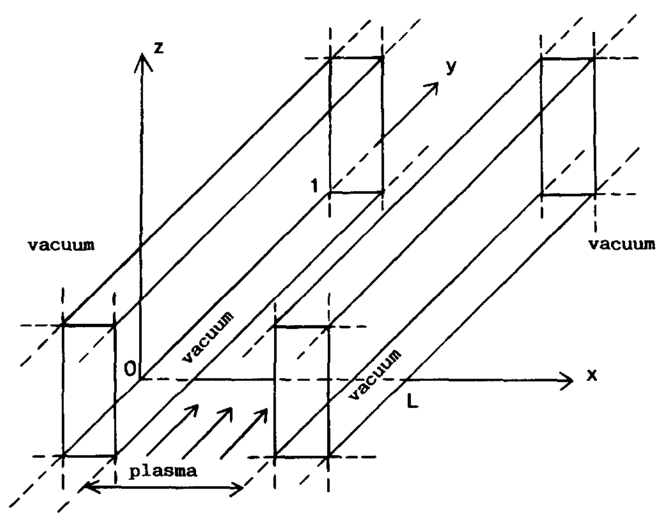

FIG. 1. Geometry of the model.

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In order to analyse the numerical stability of the scheme we shall use to solve this system, we first need to linearize all the equations. We shall linearize them around an equilibrium state (see [1 or 2]). We set $u=0$, $E=0$, and $n=n_0$ as the equilibrium state. After linearization we obtain

$$\partial_t n + \partial_x n_0 u = 0 \tag{1.1}$$

$$\partial_t(n_0 u) + \partial_x(V_T^2 n) = -(e/m)\, n_0 E \tag{1.2}$$

$$E = -\partial_x \varphi \tag{1.3}$$

$$\Delta\varphi = (e/\varepsilon_0) n, \tag{1.4}$$

where $n_i = n_0$. We set as a new variable, $v = n_0 u$. Since the system (1.1), (1.2) is linear and hyperbolic in the variables (n, v), all the classical numerical methods for solving this type of equations may be applied. Let us now look at these methods.

II. NUMERICAL METHODS AND STABILITY ANALYSIS

To discretize the Poisson equation (1.4), we use the classical three points finite difference scheme. For the electric field, we approach the solution by a centered difference scheme, so that the block of Eqs. (1.3) and (1.4) will be solved by a numerical scheme of order two.

For Eqs. (1.1) and (1.2) we use a classical upwind Godunov-type difference scheme applied to linear conservation laws, see [3], for example.

As we shall see below there is no way to avoid an implicit electric field in the right-hand side of the momentum conservation law. Both the stability analysis and the numerical experiments show that numerical solutions explode slowly in time if the electric field does not appear implicitly.

We let d be the approximate density and N the total number of space grid points. We calculate an approximate solution of the Euler equations, by the standard upwind difference scheme and we obtain the following conservative scheme:

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right), \qquad \text{with} \quad w_j^n = (d_j^n, v_j^n).$$

Taking into account the electrostatic force, we have

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{2\Delta x}\left\{\begin{matrix} V_T(-d_{j+1}^n + 2d_j^n - d_{j-1}^n) \\ \quad + (v_{j+1}^n - v_{j-1}^n) \\ V_T^2(d_{j+1}^n - d_{j-1}^n) \\ \quad + V_T(-v_{j+1}^n + 2v_j^n - v_{j-1}^n) \end{matrix}\right\} - \Delta t\, S_j, \tag{2.1}$$

where S_j denotes the approximated second member of Eq. (1.2) and is a convex combination of the electric field at two following time steps $S_j = (e/m)\, n_0(\alpha E_j^{n+1} + (1-\alpha) E_j^n)$, where α belongs to $[0, 1]$.

The discretized Poisson equation, $-\varphi_{j-1}^n + 2\varphi_j^n - \varphi_{j+1}^n = -(e/\varepsilon_0)\, \Delta x^2\, d_j^n$, and the electric field calculated by centered difference scheme, $E_j^n = (1/2\Delta x)(\varphi_{j-1}^n - \varphi_{j+1}^n)$, give the relation on E_j,

$$E_j^n = E_{j+1}^n + \frac{e}{\varepsilon_0}\frac{1}{2}\Delta x(d_{j+1}^n + d_j^n),$$

which leads, with the boundary conditions, to the expression

$$E_j^n = \frac{e}{\varepsilon_0}\Delta x\left[d_{j+1}^n + \cdots + d_{N-1}^n + \frac{1}{2}(d_j^n + d_N^n)\right].$$

In order to analyse the stability of the numerical scheme (2.1), we consider Fourier modes associated to our finite difference method (2.1). Thus we replace d_j^n and v_j^n respectively by $\hat{d}e^{i(kj\Delta x - n\omega\Delta t)}$ and $\hat{v}e^{i(kj\Delta x - n\omega\Delta t)}$ in (2.1). Setting $\theta = k\,\Delta x/2$ and $\beta = V_T\,\Delta t/\Delta x$ and collecting all the terms, we obtain the system

$$\begin{aligned} &\hat{d}(e^{-i\omega\Delta t} - 1 + 2\beta\sin^2\theta) + \hat{v}\left(i\frac{\Delta t}{\Delta x}\sin 2\theta\right) = 0, \\ &\hat{d}\left(ic\beta\sin 2\theta + \Delta t\,\Delta x\,\omega_0^2\frac{B_1}{\sin\theta}\right. \\ &\qquad \left.\times e^{i(N-j)\theta}\,(\alpha e^{-i\omega\Delta t} + 1 - \alpha)\right) \\ &\qquad + \hat{v}(e^{-i\omega\Delta t} - 1 + 2\beta\sin^2\theta) = 0, \end{aligned} \tag{2.2}$$

where $B_1 = \sin(N-j)\theta\cos\theta$ and ω_0, the plasma frequency, satisfies $\omega_0^2 = n_0 e^2/m\varepsilon_0$.

In order to study the discrete dispersion relation, we need to set the determinant of the system (2.2) to zero, which yields $(e^{-i\omega\Delta t} - 1)^2 + 4\beta\sin^2\theta(e^{-i\omega\Delta t} - 1) + 4\beta^2\sin^4\theta + \beta^2\sin^2 2\theta - i\gamma^2 R(\alpha e^{-i\omega\Delta t} + 1 - \alpha) = 0$, where $\gamma = \omega_0\,\Delta t$ and $R = e^{i(N-j)\theta}\, B_1 2\cos\theta$. Now we use the relations

$$(e^{-i\omega\Delta t} - 1)^2 = -4\sin^2\left(\frac{\omega\,\Delta t}{2}\right)e^{-i\omega\Delta t},$$

$$\sin^2\left(\frac{\omega\,\Delta t}{2}\right) = \frac{\psi^2}{1+\psi^2},$$

$$e^{i\omega\Delta t} = \frac{1 - \psi^2 + 2i\psi}{1+\psi^2} \qquad \text{if} \quad \psi = \tan\left(\frac{\omega\,\Delta t}{2}\right).$$

We then obtain the polynomial

$$P(\psi) = (4 + i\alpha\gamma^2 R - 4\beta \sin^2\theta - T)\,\psi^2 + 2iT\psi + i\alpha\gamma^2 R - 4\beta \sin^2\theta + T = 0,$$

with $T = i(1-\alpha)\,\gamma^2 R + 4\beta \sin^2\theta - 4\beta^2 \sin^4\theta - \beta^2 \sin^2 2\theta$.

From the dispersion relation, we want to find the stability conditions of the scheme (2.1), that is, the conditions which avoid the time growth of the Fourier mode. The frequency ω may be real, imaginary, or complex. Let $\omega = \omega_R + i\omega_I$, the solution is in $e^{-\omega t} = e^{-i\omega_R t} e^{\omega_I t}$ and the scheme is stable if $\omega_I \leqslant 0$. Now

$$\psi = \psi_R + i\psi_I = \frac{1}{2}\,\frac{\sin(2\omega_R\,\Delta t) + i\,\mathrm{sh}(2\omega_I\,\Delta t)}{\cos^2(\omega_R\,\Delta t) + \mathrm{sh}^2(\omega_I\,\Delta t)};$$

thus $\omega_I \leqslant 0$ is equivalent to $\psi_I \leqslant 0$. As a first step, we look for a necessary condition ensuring that $\Sigma\psi_I \leqslant 0$.

PROPOSITION 1. *A necessary condition for stability is* $\alpha = 1$.

Proof. Let $a = (4 + i\alpha\gamma^2 R - 4\beta \sin^2\theta - T)$; the sum of the zeros of the polynomial P is given by $S = -2iT/a$. If the imaginary part of S is positive, there exists a zero ψ such that ψ_I is positive and the solution explodes in time. So we want to find a condition ensuring that S_I is nonpositive. Since

$$\frac{S}{2} = -i\frac{T\bar{a}}{|a|^2} \qquad \text{and} \qquad \frac{|a|^2}{2} S = -iT,$$

it is equivalent to find a condition such that the imaginary part of $(-iT\bar{a})$ is nonpositive.

We set $A = \mathrm{Im}(-iT\bar{a}) = -T_R a_R - T_I a_I$. Recalling that

$$T = 4\beta(1-\beta)\sin^2\theta + 2i(1-\alpha)\,\gamma^2 \times \cos^2\theta \sin(N-j)\,\theta e^{i(N-j)\theta}$$

and

$$a = 4 - 4\beta(2-\beta)\sin^2\theta + 2i(2\alpha - 1)\,\gamma^2 \times \cos^2\theta \sin(N-j)\,\theta e^{i(N-j)\theta},$$

we obtain the expression

$$\begin{aligned} A = {} & 8(1-\alpha)\,\gamma^2 \cos^2\theta \sin^2(N-j)\theta \\ & + 4(1-\alpha)(1-2\alpha)\,\gamma^4 \cos^4\theta \sin^4(N-j)\theta \\ & - 8\beta[(2-\beta)(1-\alpha) + (1-\beta)(1-2\alpha)]\,\gamma^2 \\ & \times \sin^2\theta \cos^2\theta \sin^2(N-j)\theta \\ & - 16\beta(1-\beta)\sin^2\theta + 16\beta^2(1-\beta)(2-\beta)\sin^4\theta \\ & + (1-\alpha)(1-2\alpha)\,\gamma^4 \cos^4\theta \sin^2 2(N-j)\theta. \end{aligned}$$

Let the wave number k be nonvanishing and θ_0 such that, choosing j equal to $N/2$, we have $N\theta/2 = \theta_0$ fixed with $S_0 = \sin\theta_0 \neq 0$. Let $C_0 = \cos\theta_0$. We expand around $\theta = 0$, where $\theta = k\,\Delta x/2 = 2\theta_0/N$. Using the Landau notations, we obtain

$$\begin{aligned} A = {} & 8(1-\alpha)\,S_0^2\gamma^2(1 - \theta^2 + O(\theta^4)) \\ & + 4(1-\alpha)(1-2\alpha)\,S_0^4\gamma^4(1 - 4\theta^2 + O(\theta^4)) \\ & - 8\beta[(2-\beta)(1-\alpha) \\ & + (1-\beta)(1-2\alpha)]\,\gamma^2 S_0^2(\theta^2 + O(\theta^3)) \\ & - 16\beta(1-\beta)(\theta^2 + O(\theta^3)) \\ & + 16\beta^2(1-\beta)(2-\beta)\,O(\theta^4) \\ & + 4(1-\alpha)(1-2\alpha)\,S_0^2 C_0^2\gamma^4(1 - 4\theta^2 + O(\theta^4)), \end{aligned}$$

so that $A = S_0^2\gamma^2(8(1-\alpha) + 4(1-\alpha)(1-2\alpha)\gamma^2) + O(\theta^2)$.

Let $g(x) = (1-x)(1-2x)$. For x belonging to $[0, 1]$, $g(x) \geqslant -\frac{1}{8}$; therefore

$$A \geqslant S_0^2\gamma^2(8(1-\alpha) - \tfrac{1}{2}\gamma^2) + O(\theta^2).$$

If $\alpha \neq 1$, we can find Δt small enough such that $0 < \gamma^2 < 16(1-\alpha)$, and A is positive near zero. So, necessarily, $\alpha = 1$.

Remark. If we set $\alpha = 1$ in the expression of A, we obtain

$$A = 8\beta(1-\beta)\sin^2\theta \times [\gamma^2 \cos^2\theta \sin^2(N-j)\theta + 2\beta(2-\beta)\sin^4\theta - 2].$$

Let D be the term between brackets. As we expect A to be nonpositive, we want D to be nonpositive. A rough upper bound of D is $\gamma^2 + 2\beta(2-\beta) - 2$. So a necessary condition to assert that A is nonpositive, is

$$\gamma^2 + 2\beta(2-\beta) \leqslant 2. \tag{2.3}$$

We suppose from now on that $\alpha = 1$. Our goal is to derive an algebraic sufficient stability condition. We extend the Euler variables n and v to the real axis by periodicity and we apply the discrete spatial Fourier transform to the grid function $w = (w_j)$ thus obtained. As in [5], we define the discrete Fourier transform as

$$\hat{w}(k\,\Delta x) = \sum_{j=1}^{N} w_j e^{ijk\,\Delta x}$$

with

$w_j = (d_j, v_j)$ and N such that $N\,\Delta x$ is equal to the period L.

Thus we obtain the system with the same notations as in previous sections,

$$\hat{w}^{n+1}(k\,\Delta x) = G(k\,\Delta x)\,\hat{w}^n(k\,\Delta x) - \Delta t\,\Delta x\,\omega_0^2 S(k\,\Delta x),$$

with $G(k\,\Delta x) = \begin{pmatrix} a & -ib \\ -ic & a \end{pmatrix}$ and $S(k\,\Delta x) = (0, \sum_{j=1}^{N} h_j^{n+1} e^{ijk\,\Delta x})$, where $h_j^{n+1} = \frac{1}{2} d_j^{n+1} + d_{j+1}^{n+1} + \cdots + d_{N-1}^{n+1} + \frac{1}{2} d_N^{n+1}$, for $1 \leqslant j \leqslant N-1$, $h_N^{n+1} = 0$, $a = 1 - 2\beta \sin^2\theta$, $b = (\Delta t/\Delta x)\sin 2\theta$, and $c = \beta^2 \sin 2\theta$, with $\theta = k\,\Delta x/2$ and $\beta = V_T\,\Delta t/\Delta x$.

Proposition 2. *An algebraic sufficient stability condition for the scheme* (2.1) *with* $\alpha = 1$ *is*

$$\textit{CFL condition } \beta < 1 \textit{ and } L\omega_0^2\,\Delta t < 1. \qquad (2.4)$$

Proof. We would like to obtain an estimate such as $\|\hat{W}^{n+1}\| \leqslant (1 + C\,\Delta t)\,\|\hat{W}^n\|$, which is a sufficient stability condition if C is independant of time and space steps (see [5]). First, we look for an upper bound of $\|G\|$. A straightforward calculation gives $\|G\| \leqslant a^2 + \max(b^2, c^2) + |ab| + |ac|$. Since the CFL condition is satisfied, $|a| \leqslant 1$. Next, $|b| = \Delta t\,k\,|\sin 2\theta/2\theta| \leqslant k\,\Delta t$.

Let $\lambda_0 = V_T/\omega_0$ which corresponds to a Debye length, since in our framework the plasma approximation is valid, $k\lambda_0 \ll 1$ (see [1]). Hence, $|b| \leqslant (1/\lambda_0)\,\Delta t$ and $|ab| \leqslant |b| \leqslant (1/\lambda_0)\,\Delta t$. Next $|ac| \leqslant |c| \leqslant V_T^2(\Delta t/\Delta x)^2\,|\sin 2\theta| \leqslant V_T^2(k/\Delta x)\,\Delta t^2\,|\sin 2\theta/2\theta| \leqslant k\lambda_0\omega_0\,\Delta t\,\beta \leqslant \omega_0\,\Delta t$. Thus $\|G\| \leqslant 1 + c_1\,\Delta t$, with $c_1 = \max((\omega_0/2 + 1/2\lambda_0), \max(1/\lambda_0, \omega_0))$.

Now, we look for an upper bound of $\|S(k\,\Delta x)\|$. We can write $S(\sigma) = \frac{1}{2}\sum_{j=1}^{N} d_j^{n+1} e^{ij\sigma} + B$, with $\sigma = k\,\Delta x$ and $B = \sum_{j=1}^{N-1} \sum_{k=1}^{N-j} c_{j+k} d_{j+k}^{n+1} e^{ij\sigma}$, where $c_m = 1$, for $m \leqslant N-1$ and $c_N = \frac{1}{2}$.

After two changes of index ($k' = N - j - k$, $j' = j - 1$), we obtain $B = e^{i\sigma} \sum_{j=0}^{N-2} \sum_{k=0}^{N-j-2} c_{N-k} d_{N-k}^{n+1} e^{ij\sigma}$. Now, we make an inversion of summation, and we obtain $B = e^{i\sigma} \sum_{k=0}^{N-2} \sum_{j=0}^{N-k-2} c_{N-k} d_{N-k}^{n+1} e^{ij\sigma}$. We replace the sum on k by a sum on $k' = N - k$ and then the sum on j by a sum on $j' = k' - j$. Then again making an inversion of summation, we obtain $B = e^{i\sigma} \sum_{j=2}^{N} e^{-ij\sigma} \sum_{k=j}^{N} c_k d_k^{n+1} e^{ik\sigma}$. Now, $\|B\| \leqslant \sum_{j=2}^{N} \|e^{-ij\sigma} \sum_{k=j}^{N} c_k d_k^{n+1} e^{ik\sigma}\| \leqslant \sum_{j=2}^{N} (\sum_{k=j}^{N} |c_k d_k^{n+1}|^2)^{1/2} \leqslant (N-1)\,\|d^{n+1}\|$, since $c_k \leqslant 1$. Thus, $\|B\| \leqslant (N-1)\,\|\hat{W}^{n+1}\|$ and $\|S(k\,\Delta x)\| \leqslant c_2\,\Delta t\,\|\hat{W}^{n+1}\|$, where $c_2 = L\omega_0^2$. Hence $(1 - c_2\,\Delta t)\,\|\hat{W}^{n+1}\| \leqslant (1 + c_1\,\Delta t)\,\|\hat{W}^n\|$.

Because of condition (2.4), there exists ε such that $0 < \varepsilon < 1$ and $c_2\,\Delta t \leqslant 1 - \varepsilon$. Let K be such that $K \geqslant (c_1 + c_2)/\varepsilon$, we obtain $\|\hat{W}^{n+1}\| \leqslant (1 + K\,\Delta t)\,\|\hat{W}^n\|$. As c_1 and c_2 do not depend on k, Δx, and Δt, K is independant of k, Δx, and Δt.

III. NUMERICAL RESULTS

We now compare between different values of α in the nonlinear case. We can thus observe numerical stability or instability, depending on the way in which the electric field occurs in the second member of the Euler equations. Since we need to normalize all the equations, we chose the dimensionless variables

$$n' = n/n_0, \qquad u' = u/c, \qquad t' = \omega_0 t, \qquad x' = x\omega_0/c,$$

with $\omega_0 = k_0 c = n_0 e^2/m\varepsilon_0$, where c is the light speed.

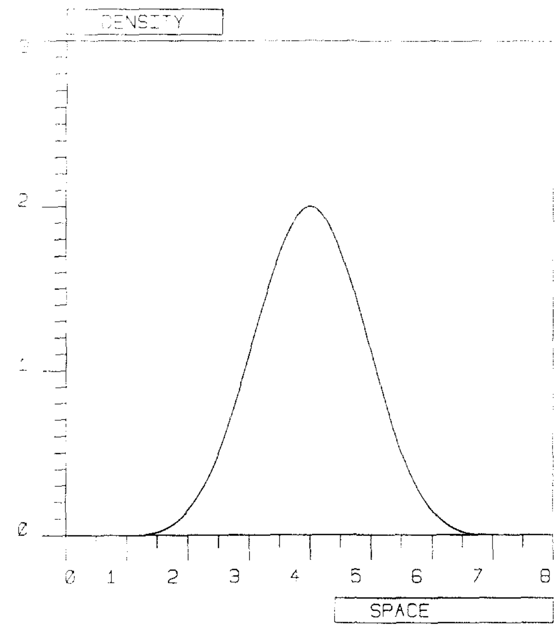

FIG. 2. Initial electronic density.

In the same way we shall denote the dimensioneless variables and the variables with dimension for the sake of simplicity. Our nonlinear equations, written here without dimensions, are

$$\partial_t n + \partial_x nu = 0, \qquad (3.1)$$

$$\partial_t nu + \partial_x(nu^2 + n\hat{c}^2) = -nE, \qquad (3.2)$$

$$E = -\nabla\varphi, \qquad (3.3)$$

$$-\Delta\varphi = (n_i - n), \qquad (3.4)$$

with the same notations as before and the dimensionless speed $\hat{c}$ verifies $\hat{c}^2 = KT/mc^2$, where c is the light speed.

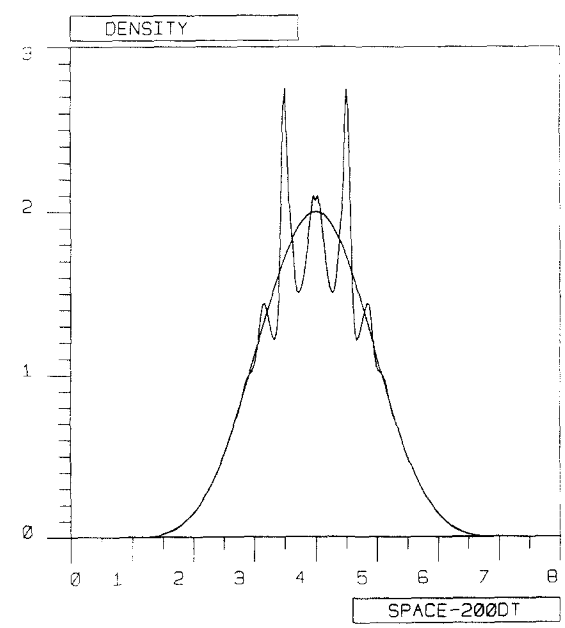

Fig. 3. Space evolution of the electronic density for $\alpha = 0$, after $200\,\Delta t$.

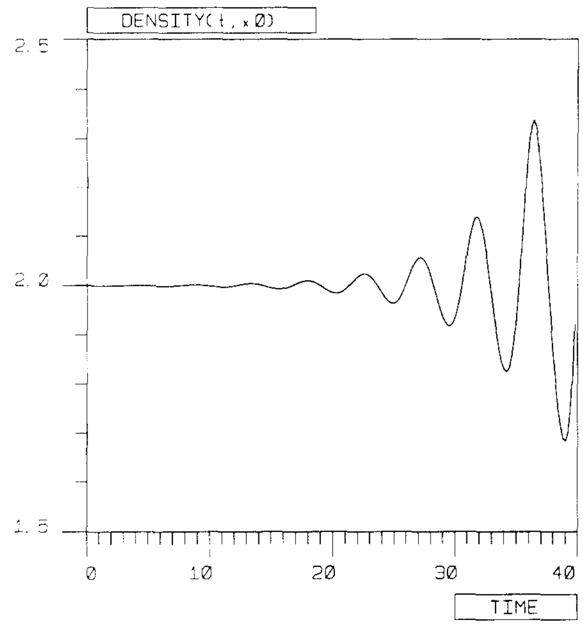

FIG. 4. Time evolution of the electronic density for $\alpha = 0$.

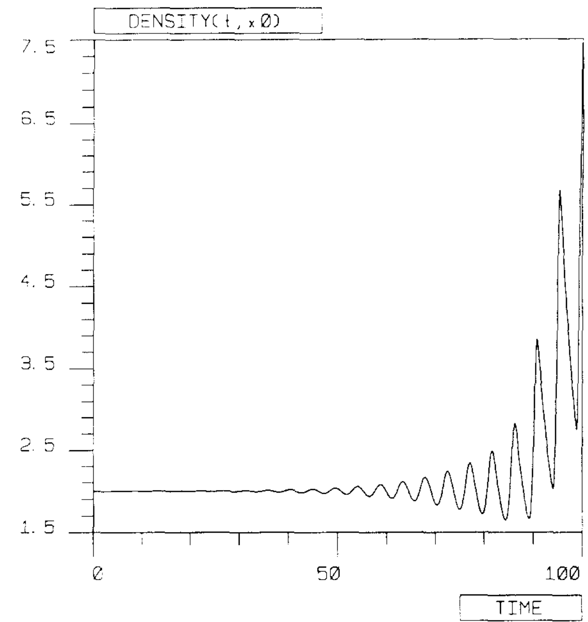

FIG. 6. Time evolution of the electronic density for $\alpha = \frac{1}{2}$.

There are many numerical methods to solve the Euler equations in the nonlinear case though some are more expensive than others. After having tested several kinds, we have chosen the Van Leer flux-vector splitting, because it is almost as precise as the others and cheaper. The general idea of this method is the same as the usual upwind difference scheme, but for more details see [4]. It consists in splitting the flux function into two functions, a forward flux and a backward flux, the gradients of which will correspond to positive and negative speeds of propagation. Letting $g_{j+1/2}$ be the Van Leer flux calculated at the two cells j and $j+1$ interface, we obtain the scheme to solve Eqs. (3.1) and (3.2),

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right) - \begin{Bmatrix} 0 \\ \Delta t\left(\alpha d_j^{n+1} E_j^{n+1} + (1-\alpha)\, d_j^n E_j^n\right) \end{Bmatrix},$$

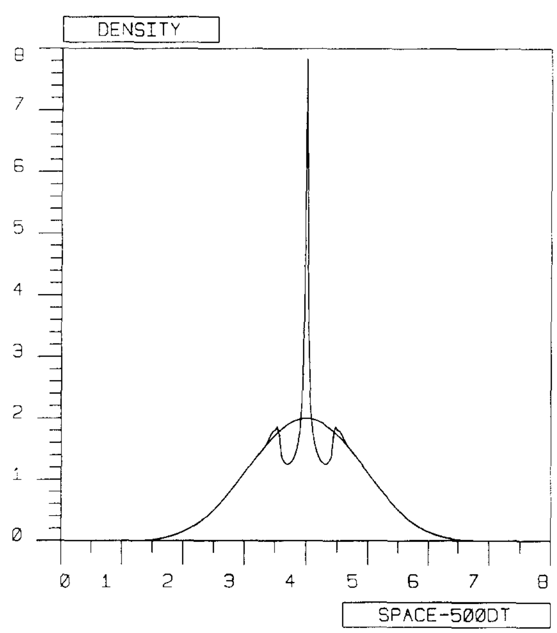

FIG. 5. Space evolution of the electronic density for $\alpha = 1/2$, after 500 Δt.

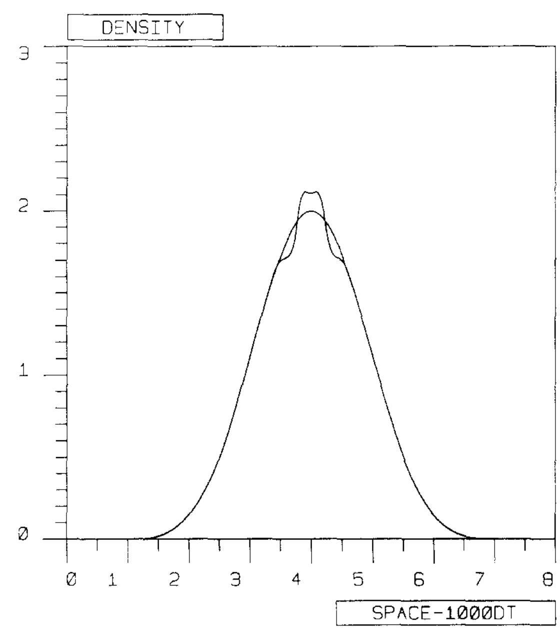

FIG. 7. Space evolution of the electronic density for $\alpha = \frac{3}{4}$, after 1000 Δt.

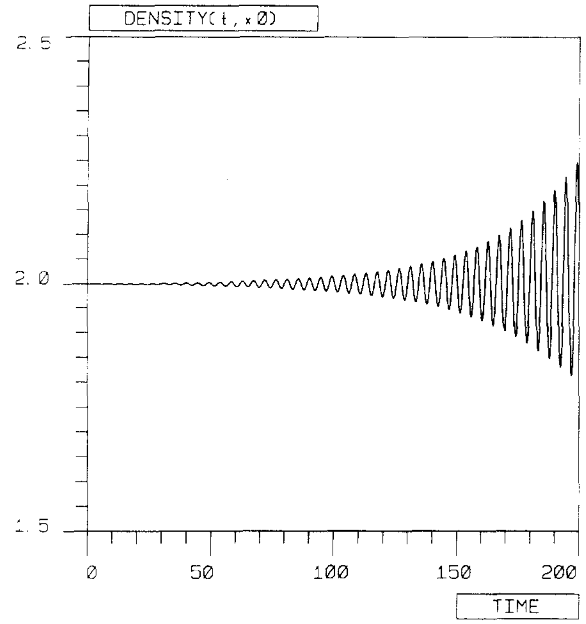

FIG. 8. Time evolution of the electronic density for $\alpha = \frac{3}{4}$.

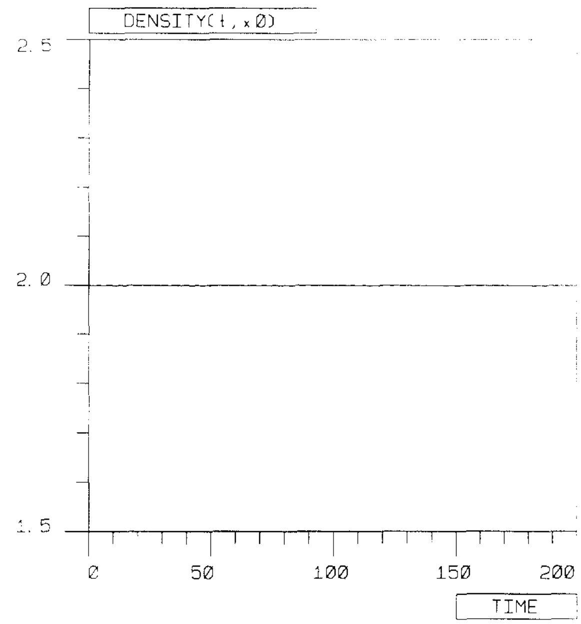

FIG. 10. Time evolution of the electronic density for $\alpha = 1$.

where the electric field is calculated in the same way as before. Since the Van Leer decomposition involves the Mach number and since the Mach number cannot be calculated if the density vanishes (because u is calculated by nu/n), we add numerical threshold on density and momentum quantity. We have imposed density and momentum quantity to vanish if the density is smaller than 10^{-7} (10^{-7} is often the computer precision). Hence the flux vanishes too.

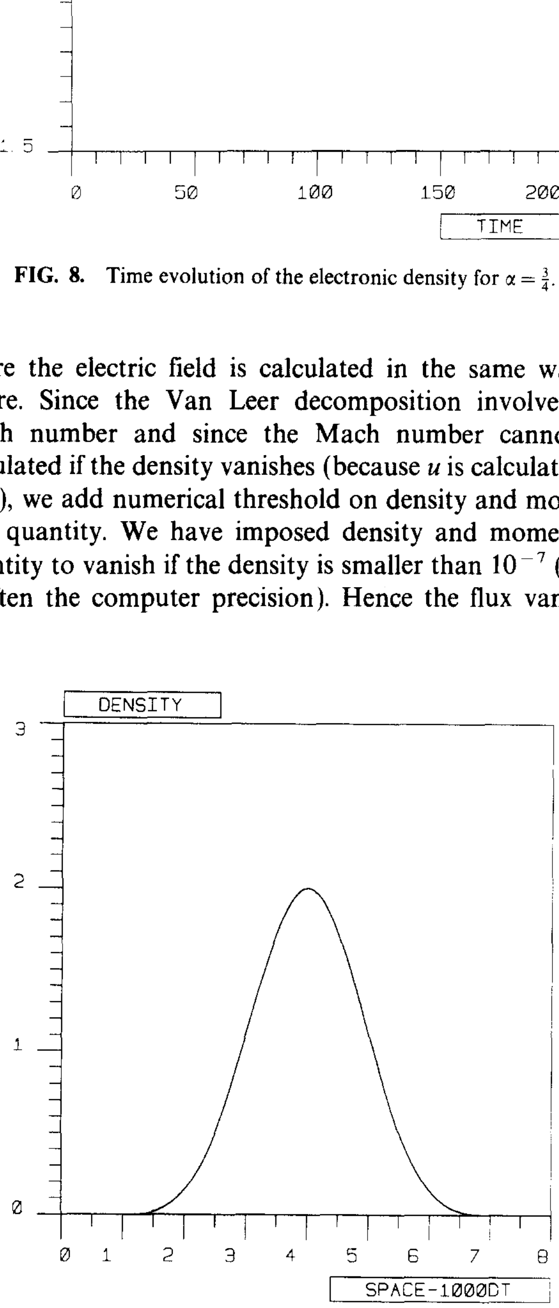

FIG. 9. Space evolution of the electronic density for $\alpha = 1$, after $1000\,\Delta t$.

Keeping in mind the stability analysis of the associated linearized problem, we expect stability only if $\alpha = 1$. The initial conditions are the following: the speed u and the electric field are equal zo zero and the density is a cubic spline (see Fig. 2). We use 240 grid points in space uniformly distributed in $[0, L]$ with $L = 16\pi$, and assume that Δt is equal to 0.2 ($\omega_0 = 1$ by normalization). We set $KT = 4$ KeV so that $\hat{c}^2 = 8.10^{-3}$. We then take 30 points by wave length, so that $\Delta x = 2\pi/30$. Notice that with these values the condition (2.3) is satisfied. Figures 3, 5, 7, and 9 show the space evolution of the electronic density together with the initial density, for $\alpha = 0, \frac{1}{2}, \frac{3}{4}, 1$, respectively. Figures 4, 6, 8, 10 show the time evolution of the electronic density at a fixed space point with $\alpha = 0, \frac{1}{2}, \frac{3}{4}, 1$, respectively. We have chosen this space point at the maximum density to make sure that the fluid velocity is very small, so that we are very close to the linearization conditions. If α is not equal to 1, we can see that the closer α is to 1, the slower the instability appears; it is asymptotically unstable and we had to increase the number of time steps to check that when α is to close to 1 (we have tried $\alpha = 1 - \varepsilon$ with small ε).

CONCLUSION

Theoretical analysis and numerical experiments show that the only stable method used to solve the Euler–Poisson equations proposed here, is the one where the electric field is treated fully implicitely in the second member of the momementum conservation law. As said in the Introduction, a particles method used to solve the collisionless

Boltzmann equation coupled with the Poisson equation in a similar physical parameters range does not impose such a strong stability condition. The point in that case is that the scheme is applied for each value of the fluid velocity u and that u is constant for each beam describing the distribution function. The electric field may indeed be introduced, in those methods, semi-implicitly at time $n+\frac{1}{2}$; see [2, 6]. So, here we can see another difference between fluid numerical methods and particle numerical methods used in plasma physics.

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